

A Bi-Hamiltonian Formulation for Triangular Systems by Perturbations

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Abstract

A bi-Hamiltonian formulation is proposed for triangular systems resulted by perturbations around solutions, from which infinitely many symmetries and conserved functionals of triangular systems can be explicitly constructed, provided that one operator of the Hamiltonian pair is invertible. Through our formulation, four examples of triangular systems are exhibited, which also show that bi-Hamiltonian systems in both lower dimensions and higher dimensions are many and varied. Two of four examples give local $2 + 1$ dimensional bi-Hamiltonian systems and illustrate that multi-scale perturbations can lead to higher-dimensional bi-Hamiltonian systems.

1 Introduction

The bi-Hamiltonian formulation is a great success in the Hamiltonian theory of differential equations [1]. It has attracted the attention of a wide audience within both the mathematical community and the physical community due to its importance in producing symmetries and conserved functionals, and has already become one of active research directions in the field of soliton theory and integrable systems.

In this paper, we are concerned with the bi-Hamiltonian formulation of triangular systems resulted by various perturbations around solutions, specific systems of which were furnished in [2, 3, 4]. Such triangular systems provide candidates of integrable couplings for given integrable systems [5, 6]. A general triangular system reads as

$$\begin{cases} u_t = K(u) = K(u, \dots, u^{(k)}), \\ v_t = S(u, v) = S(u, v, \dots, u^{(l)}, v^{(l)}), \end{cases} \quad (1.1)$$

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where $u = u(t, x)$, $v = v(t, x)$, and $u^{(n)}$ and $v^{(n)}$ are derivatives with respect to the spatial variable x . Such a concrete example by a first-order perturbation is given by

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \end{cases} \quad (1.2)$$

where $K'(u)[v]$ denotes the Gateaux derivative of $K(u)$ at a direction v , i.e.,

$$K'(u)[v] = \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} K(u + \varepsilon v).$$

A mathematical structure called the perturbation bundle has been established in [5] to study its integrable properties. Note that the second component of the above system is just the linearized system of the original system $u_t = K(u)$, and thus the symmetry problem leads to a triangular system together with the original system, which also shows the importance of studying triangular systems (see [6] for more discussion). Other similar examples of specific triangular systems were presented by means of perturbations in [2, 3, 4]. However, there is no discussion on the bi-Hamiltonian formulation of general triangular systems by perturbations, even the specific systems mentioned above.

With a view to exposing integrability, we would like to answer whether there exists any bi-Hamiltonian formulation for triangular systems resulted by various perturbations around solutions. It will be shown that a bi-Hamiltonian formulation of the resulting triangular systems can be inherited from an original bi-Hamiltonian system. The general formulation allows us to present various examples of bi-Hamiltonian systems, in both $1 + 1$ and $2 + 1$ dimensions.

The paper is organized as follows. In Section 2, we shall choose perturbed systems of given bi-Hamiltonian systems as starting systems and introduce our triangular systems by using perturbations around solutions of starting systems, which contain many special triangular systems in [2, 3, 4]. Then in Section 3, a bi-Hamiltonian formulation for the resulting triangular systems will be proposed, based on the bi-Hamiltonian formulation of starting systems. In Section 4, we will go on to exhibit four examples of triangular systems through the general bi-Hamiltonian formulation, which also show that bi-Hamiltonian systems in both $1 + 1$ and $2 + 1$ dimensions are many and varied. Two of four examples give local $2 + 1$ dimensional bi-Hamiltonian systems and illustrate that multi-scale perturbations can lead to higher-dimensional bi-Hamiltonian systems. Finally in Section 5, some concluding remarks will be given.

2 Triangular systems by perturbations

2.1 Bi-Hamiltonian systems:

Assume that we have a bi-Hamiltonian system

$$u_t = K(u) = J \frac{\delta \tilde{H}_1}{\delta u} = M \frac{\delta \tilde{H}_0}{\delta u}, \quad \tilde{H}_0 = \int H_0 dx, \quad \tilde{H}_1 = \int H_1 dx, \quad (2.1)$$

where J and M constitute a Hamiltonian pair (see [7, 8, 9] for more information), t is a single variable but x can be a single or vector variable. If one operator of the Hamiltonian pair is invertible, we can have infinitely many symmetries $\{K_n\}_{n=0}^\infty$ and conserved functionals $\{\tilde{H}_n\}_{n=0}^\infty$, which can be explicitly computed through

$$\begin{cases} K_n = \Phi^{n-1} K(u) = (MJ^{-1})^{n-1} K(u), & n \geq 1, \\ \tilde{H} = \int H_n dx, & H_n = \int_0^1 \langle u, (J^{-1} \Phi^{n-1} K)(\lambda u) \rangle d\lambda, & n \geq 0, \end{cases} \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of the corresponding Euclidean space. Moreover, they are related through the bi-Hamiltonian formulation [10]

$$K_n = J \frac{\delta \tilde{H}_n}{\delta u} = M \frac{\delta \tilde{H}_{n-1}}{\delta u}, \quad n \geq 1. \quad (2.3)$$

Fuchssteiner and Fokas [11] discovered an important fact that when J and M constitute a Hamiltonian pair and J is invertible, the operator $\Phi = MJ^{-1}$ is hereditary [12], i.e.,

$$\Phi'[\Phi X]Y - \Phi\Phi'[X]Y = \Phi'[\Phi Y]X - \Phi\Phi'[Y]X$$

holds for any vector fields X and Y , where $\Phi'[X]$ denotes the Gateaux derivative

$$\Phi'[X] = \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \Phi(u + \varepsilon X).$$

This condition is actually equivalent to an invariance of Lie derivative of Φ (see, for example, [13, 14]). It is the hereditaryness of Φ that gives rise to an explanation why soliton systems come in hierarchies.

2.2 Perturbed systems:

Let us now choose a perturbed system with a perturbation parameter ε :

$$u_t = K^{\text{per}}(u) := \sum_{j=0}^m \alpha_j \varepsilon^j K_{i_j}(u) = J \frac{\delta \tilde{H}_1^{\text{per}}}{\delta u} = M \frac{\delta \tilde{H}_0^{\text{per}}}{\delta u}, \quad (2.4)$$

where $m \geq 0$, the α_j are arbitrary constants, the i_j are arbitrary natural numbers (which means to take arbitrary vector fields $K_{i_j}(u)$ from $\{K_n\}_{n=1}^\infty$), and two Hamiltonian functionals read

$$\tilde{H}_0^{\text{per}} = \int H_0^{\text{per}} dx, \quad H_0^{\text{per}} = \sum_{j=0}^m \alpha_j \varepsilon^j H_{i_j}, \quad \tilde{H}_1^{\text{per}} = \int H_1^{\text{per}} dx, \quad H_1^{\text{per}} = \sum_{j=0}^m \alpha_j \varepsilon^j H_{i_j+1}. \quad (2.5)$$

This system (2.4) is called a starting system, which is nothing but a generalized system of the original bi-Hamiltonian system (2.1). Following the general scheme shown in (2.2), we have infinitely many symmetries and conserved functionals

$$\begin{cases} K_n^{\text{per}} := \Phi^{n-1} K^{\text{per}} = \sum_{j=0}^m \alpha_j \varepsilon^j K_{i_j+n-1}(u), & n \geq 1, \\ \tilde{H}_n^{\text{per}} := \int H_n^{\text{per}} dx, & H_n^{\text{per}} = \sum_{j=0}^m \alpha_j \varepsilon^j H_{i_j+n}(u), & n \geq 0, \end{cases} \quad (2.6)$$

for the starting system (2.4), since we can directly check that

$$K_n^{\text{per}} = J \frac{\delta \tilde{H}_n^{\text{per}}}{\delta u} = M \frac{\delta \tilde{H}_{n-1}^{\text{per}}}{\delta u}, \quad n \geq 1.$$

2.3 Triangular systems:

For any integers $N \geq 0$ and $r \geq 0$, take a perturbation series:

$$\hat{u}_N = \sum_{i=0}^N \varepsilon^i \eta_i, \quad \eta_i = \eta_i(y_0, y_1, y_2, \dots, y_r, t), \quad (2.7)$$

where $y_i = \varepsilon^i x$, $0 \leq i \leq N$, are all slow variables. Now we make a perturbation around solutions of the starting system (2.4) and observe the N -th order perturbation system

$$\hat{u}_{Nt} = K^{\text{per}}(\hat{u}_N) + o(\varepsilon^N),$$

where u, η_i , $0 \leq i \leq N$, are supposed to be column vectors of the same dimension. By the Taylor expansion, this leads to an equivalent and bigger system

$$\eta_{it} = \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} K^{\text{per}}(\hat{u}_N) \Big|_{\varepsilon=0} = \sum_{j=0}^{\min(m,i)} \frac{\alpha_j}{(i-j)!} \frac{\partial^{i-j}}{\partial \varepsilon^{i-j}} K_{i_j}(\hat{u}_N) \Big|_{\varepsilon=0}, \quad 0 \leq i \leq N. \quad (2.8)$$

For brevity, we rewrite it as a concise form

$$\hat{\eta}_{Nt} = (\text{per}_N K^{\text{per}})(\hat{\eta}_N) = ((\text{per}_N K^{\text{per}})_0^T, \dots, (\text{per}_N K^{\text{per}})_N^T)^T, \quad \hat{\eta}_N = (\eta_0^T, \dots, \eta_N^T)^T, \quad (2.9)$$

where T denotes the matrix transpose. Noting that

$$\hat{u}_N = \hat{u}_i + \varepsilon^{i+1} \sum_{j=0}^{N-i-1} \varepsilon^j \eta_{j+i+1}, \quad \hat{u}_i = \sum_{j=0}^i \varepsilon^j \eta_j, \quad 0 \leq i \leq N-1,$$

an application of the Taylor expansion tells us that

$$(\text{per}_N K^{\text{per}})_i = \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} K^{\text{per}}(\hat{u}_N) \Big|_{\varepsilon=0} = \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} K^{\text{per}}(\hat{u}_i) \Big|_{\varepsilon=0}, \quad 0 \leq i \leq N-1,$$

and thus the perturbation system (2.8) is triangular, i.e., the $(i + 1)$ -th component $\eta_{it} = (\text{per}_N K^{\text{per}})_i$ just involves the first $i + 1$ dependent variables η_0, \dots, η_i but not the other dependent variables $\eta_{i+1}, \dots, \eta_N$.

In our formulation, the superscript “per” denotes the perturbed objects such as the perturbed tensor fields and the perturbed functionals as in (2.4) and (2.5), but the prefix “per_N” means the perturbation resulting from the N -th order perturbation (2.7) of the dependent variable u . The small parameter ε is involved in both the starting system (2.4) and the perturbation series (2.7), but there is no relation among three integers m , N , and r that we need to take in the starting system (2.4) and the perturbation series (2.7). This demonstrates diversity to formulate our triangular systems. If we take a special choice of $\alpha_0 = 1$ and $K_{i_0} = K$ in our construction, the triangular system (2.8) becomes a coupling system of $u_t = K(u)$, because its first component is $\eta_{0t} = K(\eta_0)$. This paves a way for constructing integrable couplings of given integrable systems [4, 6]. If a starting system is particularly chosen as

$$u_t = K^{\text{per}}(u) = K(u) + \alpha\varepsilon K(u), \quad \alpha = \text{const.}, \quad (2.10)$$

the following specific triangular system

$$\begin{cases} \eta_{0t} = K(\eta_0), \\ \eta_{1t} = K'(\eta_0)[\eta_1] + \alpha K(\eta_0), \end{cases} \quad (2.11)$$

will be engendered upon making a first-order perturbation. This system looks simple, but it generalizes the triangular system (1.2) resulting from the symmetry problem. The main objective of this paper is to propose a bi-Hamiltonian formulation for the triangular systems determined by (2.8), which contain two specific interesting triangular systems (1.2) and (2.11).

3 Bi-Hamiltonian formulation

For now on, we focus on the establishment of a bi-Hamiltonian formulation for the triangular systems determined by (2.8). We would actually like to show that a bi-Hamiltonian formulation of the resulting triangular systems can be inherited from an original bi-Hamiltonian system (2.1).

To the end, let us first introduce a new Hamiltonian pair:

$$(\text{per}_N J)(\hat{\eta}_N) \equiv \hat{J}_N(\hat{\eta}_N) \quad \text{and} \quad (\text{per}_N M)(\hat{\eta}_N) \equiv \hat{M}_N(\hat{\eta}_N), \quad (3.1)$$

which are defined as follows

$$\begin{aligned}
(\text{per}_N P)(\hat{\eta}_N) &\equiv \hat{P}_N(\hat{\eta}_N) = \left[(\hat{P}_N(\hat{\eta}_N))_{ij} \right]_{(N+1) \times (N+1)} \\
&= \left[\frac{1}{(i+j-N)!} \frac{\partial^{i+j-N} P(\hat{u}_N)}{\partial \varepsilon^{i+j-N}} \Big|_{\varepsilon=0} \right]_{(N+1) \times (N+1)} \\
&= \begin{bmatrix} 0 & \cdots & 0 & P(\eta_0) \\ \vdots & \ddots & \ddots & \frac{1}{1!} \frac{\partial P(\hat{u}_N)}{\partial \varepsilon} \Big|_{\varepsilon=0} \\ 0 & \ddots & \ddots & \vdots \\ P(\eta_0) & \frac{1}{1!} \frac{\partial P(\hat{u}_N)}{\partial \varepsilon} \Big|_{\varepsilon=0} & \cdots & \frac{1}{N!} \frac{\partial^N P(\hat{u}_N)}{\partial \varepsilon^N} \Big|_{\varepsilon=0} \end{bmatrix}, \quad P = J, M; \tag{3.2}
\end{aligned}$$

and a new hereditary recursion operator defined by

$$\begin{aligned}
(\text{per}_N \Phi)(\hat{\eta}_N) &\equiv \hat{\Phi}_N(\hat{\eta}_N) = \left[(\hat{\Phi}_N(\hat{\eta}_N))_{ij} \right]_{(N+1) \times (N+1)} \\
&= \left[\frac{1}{(i-j)!} \frac{\partial^{i-j} \Phi(\hat{u}_N)}{\partial \varepsilon^{i-j}} \Big|_{\varepsilon=0} \right]_{(N+1) \times (N+1)} \\
&= \begin{bmatrix} \Phi(\eta_0) & 0 & \cdots & 0 \\ \frac{1}{1!} \frac{\partial \Phi(\hat{u}_N)}{\partial \varepsilon} \Big|_{\varepsilon=0} & \Phi(\eta_0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{N!} \frac{\partial^N \Phi(\hat{u}_N)}{\partial \varepsilon^N} \Big|_{\varepsilon=0} & \cdots & \frac{1}{1!} \frac{\partial \Phi(\hat{u}_N)}{\partial \varepsilon} \Big|_{\varepsilon=0} & \Phi(\eta_0) \end{bmatrix}; \tag{3.3}
\end{aligned}$$

where $i, j = 0, 1, \dots, N$, and \hat{u}_N is defined by (2.7). These operators will be used to establish our new bi-Hamiltonian formulation for the triangular systems by (2.8). The structures of these operators originate from those proposed for single Hamiltonian formulations in [4]. Only a difference is the scale of perturbations. In our previous work [4], single-scale perturbations were considered, but in this paper, multi-scale perturbations will need to be considered. Like single-scale perturbations, multi-scale perturbations also guarantees that two operators defined by (3.2) constitute a Hamiltonian pair and the operator defined by (3.3) is hereditary. The proofs are direct and very similar to those in the case of single-scale perturbations [4], although they are rather laborious and much harder (see [6] for a detailed proof). Obviously, however, new operators still satisfy the following coupled condition:

$$\hat{\Phi}_N = \hat{M}_N(\hat{J}_N)^{-1}, \quad \hat{J}_N \hat{\Psi}_N = \hat{\Phi}_N \hat{J}_N, \quad \hat{\Psi}_N = (\hat{\Phi}_N)^\dagger, \tag{3.4}$$

where the superscript \dagger means to take the adjoint operation. The existence of the inverse operator $(\hat{J}_N)^{-1}$ is guaranteed by the existence of J^{-1} , following the definition of \hat{J}_N^{-1} as in (3.1) and (3.2). The coupled condition (3.4) ensures [1] that conserved functionals recursively determined by $\hat{\Phi}_N$ commute with each other under two Poisson brackets generated by \hat{J}_N and \hat{M}_N .

For the triangular system defined by (2.8), new Hamiltonian functionals can be chosen as

$$(\text{per}_N \tilde{H}_0^{\text{per}})(\hat{\eta}_N) := \frac{1}{N!} \frac{\partial^N}{\partial \varepsilon^N} \tilde{H}_0^{\text{per}}(\hat{u}_N) \Big|_{\varepsilon=0} = \sum_{j=0}^{\min(m,N)} \frac{\alpha_j}{(N-j)!} \frac{\partial^{N-j}}{\partial \varepsilon^{N-j}} \tilde{H}_{i_j}(\hat{u}_N) \Big|_{\varepsilon=0}, \quad (3.5)$$

$$(\text{per}_N \tilde{H}_1^{\text{per}})(\hat{\eta}_N) := \frac{1}{N!} \frac{\partial^N}{\partial \varepsilon^N} \tilde{H}_1^{\text{per}}(\hat{u}_N) \Big|_{\varepsilon=0} = \sum_{j=0}^{\min(m,N)} \frac{\alpha_j}{(N-j)!} \frac{\partial^{N-j}}{\partial \varepsilon^{N-j}} \tilde{H}_{i_j+1}(\hat{u}_N) \Big|_{\varepsilon=0}, \quad (3.6)$$

where \tilde{H}_0^{per} and \tilde{H}_1^{per} are defined by (2.5), and \hat{u}_N is defined by (2.7). They will offer the required Hamiltonian functionals in our bi-Hamiltonian formulation of the triangular system (2.8). The above crucial form of the Hamiltonian functionals are motivated by a study of the perturbation system of the KdV equation [16].

Now a direct computation can show that the triangular system (2.8) has the following bi-Hamiltonian formulation

$$\hat{\eta}_{Nt} = \hat{J}_N \frac{\delta(\text{per}_N \tilde{H}_1^{\text{per}})}{\delta \hat{\eta}_N} = \hat{M}_N \frac{\delta(\text{per}_N \tilde{H}_0^{\text{per}})}{\delta \hat{\eta}_N}, \quad \hat{\eta}_N = (\eta_0^T, \eta_1^T, \dots, \eta_N^T)^T. \quad (3.7)$$

Here a Hamiltonian pair of \hat{J}_N and \hat{M}_N is defined by (3.1) and (3.2), and two Hamiltonian functionals $\text{per}_N \tilde{H}_0^{\text{per}}$ and $\text{per}_N \tilde{H}_1^{\text{per}}$ are defined by (3.5) and (3.6). The bi-Hamiltonian formulation (3.7) is what we intend to establish for the triangular system (2.8). It follows that the triangular system (2.8) is a good example of integrable systems.

In fact, let us first introduce

$$\begin{cases} \text{per}_N K_n^{\text{per}} := ((K_n^{\text{per}}(\hat{u}_N))^T \Big|_{\varepsilon=0}, \frac{1}{1!} \frac{\partial}{\partial \varepsilon} (K_n^{\text{per}}(\hat{u}_N))^T \Big|_{\varepsilon=0}, \dots, \frac{1}{N!} \frac{\partial^N}{\partial \varepsilon^N} (K_n^{\text{per}}(\hat{u}_N))^T \Big|_{\varepsilon=0})^T, & n \geq 1, \\ \text{per}_N \tilde{H}_n^{\text{per}} := \frac{1}{N!} \frac{\partial^N}{\partial \varepsilon^N} \tilde{H}_n^{\text{per}}(\hat{u}_N) \Big|_{\varepsilon=0}, & n \geq 0, \end{cases} \quad (3.8)$$

where K_n^{per} and \tilde{H}_n^{per} are defined by (2.6), and \hat{u}_N is defined by (2.7). Then it can directly be verified that

$$\begin{cases} \text{per}_N K_n^{\text{per}} = (\hat{\Phi}_N)^{n-1} (\text{per}_N K^{\text{per}}), & n \geq 1, \\ \text{per}_N \tilde{H}_n^{\text{per}} = \sum_{j=0}^{\min(m,N)} \frac{\alpha_j}{(N-j)!} \frac{\partial^{N-j}}{\partial \varepsilon^{N-j}} \tilde{H}_{i_j+n}(\hat{u}_N) \Big|_{\varepsilon=0}, & n \geq 0, \end{cases}$$

and further

$$\text{per}_N K_n^{\text{per}} = \hat{J}_N \frac{\delta(\text{per}_N \tilde{H}_n^{\text{per}})}{\delta \hat{\eta}_N} = \hat{M}_N \frac{\delta(\text{per}_N \tilde{H}_{n-1}^{\text{per}})}{\delta \hat{\eta}_N}, \quad n \geq 1.$$

Therefore, it follows from the bi-Hamiltonian formulation (3.7) that $\text{per}_N K_n^{\text{per}}$, $n \geq 1$, and $\text{per}_N(\tilde{H}_n^{\text{per}})$, $n \geq 0$, defined by (3.8), are symmetries and conserved functionals of the triangular

system (2.8), respectively. This implies that the triangular system (2.8) is integrable if we start from a bi-Hamiltonian system (2.1).

Summing up, the above manipulation shows how to inherit the bi-Hamiltonian formulation and to compute symmetries and conserved functionals for the triangular system (2.8) while taking perturbations for the starting system (2.4). In the next section, we perform applications of the above formulation to four concrete examples, in which new bi-Hamiltonian systems in both $1 + 1$ dimensions and $2 + 1$ dimensions will be formulated.

4 Illustrative examples

Let us consider the KdV equation

$$u_t = K(u) = u_{xxx} + 6uu_x. \quad (4.1)$$

It is well known that it has a local bi-Hamiltonian formulation [1, 10]

$$u_t = K(u) = J \frac{\delta \tilde{H}_1}{\delta u} = M \frac{\delta \tilde{H}_0}{\delta u}, \quad (4.2)$$

where the Hamiltonian pair and the Hamiltonian functionals are given by

$$J = \partial_x, \quad M = \partial_x^3 + 4u\partial_x + 2u_x, \quad \tilde{H}_0 = \int \frac{1}{2}u^2 dx, \quad \tilde{H}_1 = \int \left(\frac{1}{2}uu_{xx} + u^3\right) dx. \quad (4.3)$$

Therefore, it has infinitely many symmetries and conserved functionals

$$K_n = \Phi^n u_x, \quad \tilde{H}_n = \int H_n dx, \quad H_n = \int_0^1 u(\Psi^n u)(\lambda u) d\lambda, \quad n \geq 0, \quad (4.4)$$

where the hereditary recursion operator Φ and its adjoint operator Ψ read as

$$\Phi(u) = MJ^{-1} = \partial_x^2 + 4u + 2u_x\partial_x^{-1}, \quad \Psi = \Phi^\dagger = \partial_x^2 + 4u - 2\partial_x^{-1}u_x, \quad (4.5)$$

where $\partial^{-1}\partial = \partial\partial^{-1} = 1$. For example, we can obtain

$$\begin{cases} K_2(u) = u_{5x} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x, \\ H_2(u) = \frac{1}{2}uu_{4x} + \frac{10}{3}u^2u_{xx} + \frac{5}{3}uu_x^2 + \frac{5}{2}u^4. \end{cases} \quad (4.6)$$

Note that in (4.4) we added $K_0 = u_x$ to the Abelian symmetry algebra $\{K_n\}_{n=1}^\infty$ as defined in (2.2).

If we choose the original KdV equation as a starting equation, taking single-scale perturbations leads to the standard perturbation KdV systems [15], which were proved to be bi-Hamiltonian [16]. In what follows, we will formulate other examples of integrable couplings for

the KdV equation, by choosing proper perturbed equations as starting equations and taking bi-scale perturbations in the subsection 4.2. All examples also show that bi-Hamiltonian systems in both $1 + 1$ and $2 + 1$ dimensions are many and varied.

4.1 The case of single-scale perturbations:

We take a special perturbed equation

$$u_t = K^{\text{per}}(u) = K_1(u) + \varepsilon K_1(u) = J \frac{\delta \tilde{H}_1^{\text{per}}}{\delta u} = M \frac{\delta \tilde{H}_0^{\text{per}}}{\delta u} \quad (4.7)$$

with two Hamiltonian functionals

$$\tilde{H}_0^{\text{per}} = \tilde{H}_0 + \varepsilon \tilde{H}_0, \quad \tilde{H}_1^{\text{per}} = \tilde{H}_1 + \varepsilon \tilde{H}_1 \quad (4.8)$$

as a starting equation. Here $K_1 = K, J, M, \tilde{H}_0, \tilde{H}_1$ are given by (4.3), (4.4) and 4.5). The first-order perturbation

$$\hat{u}_1 = \eta_0 + \varepsilon \eta_1$$

yields the following triangular system

$$\begin{cases} \eta_{0t} = K_1(\eta_0) = \eta_{0xxx} + 6\eta_0\eta_{0x}, \\ \eta_{1t} = K_1'(\eta_0)[\eta_1] + K_1(\eta_1) = \eta_{1xxx} + 6(\eta_0\eta_1)_x + \eta_{0xxx} + 6\eta_0\eta_{0x}. \end{cases} \quad (4.9)$$

According to our scheme of construction in Section 3, its Hamiltonian pair and corresponding hereditary recursion operator are

$$\hat{J}_1 = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix}, \quad \hat{M}_1 = \begin{bmatrix} 0 & M_0 \\ M_0 & M_1 \end{bmatrix}, \quad \hat{\Phi}_1 = \begin{bmatrix} \Phi_0 & 0 \\ \Phi_1 & \Phi_0 \end{bmatrix}, \quad (4.10)$$

with the entries of \hat{M}_1 and $\hat{\Phi}_1$ being defined by

$$M_i = \delta_{i0}\partial_x^3 + 4\eta_0\partial_x + 2\eta_{0x}, \quad \Phi_i = \delta_{i0}\partial_x^2 + 4\eta_i + 2\eta_{ix}\partial_x^{-1}, \quad i = 0, 1, \quad (4.11)$$

where δ_{0i} is the Kronecker symbol. The triangular system (4.9) has a local bi-Hamiltonian formulation:

$$\hat{\eta}_{1t} = \hat{J}_1 \frac{\delta(\text{per}_1 \tilde{H}_1^{\text{per}})}{\delta \hat{\eta}_1} = \hat{M}_1 \frac{\delta(\text{per}_1 \tilde{H}_0^{\text{per}})}{\delta \hat{\eta}_1}, \quad \hat{\eta}_1 = (\eta_0, \eta_1)^T, \quad (4.12)$$

with two Hamiltonian functionals

$$\begin{cases} \text{per}_1 \tilde{H}_0^{\text{per}} = (\text{per}_1 \tilde{H}_0^{\text{per}})(\hat{\eta}_1) = \frac{\partial \tilde{H}_0(\hat{u}_1)}{\partial \varepsilon} \Big|_{\varepsilon=0} + \tilde{H}_0(\eta_0) = \int (\eta_0\eta_1 + \frac{1}{2}\eta_0^2) dx, \\ \text{per}_1 \tilde{H}_1^{\text{per}} = (\text{per}_1 \tilde{H}_1^{\text{per}})(\hat{\eta}_1) = \frac{\partial \tilde{H}_1(\hat{u}_1)}{\partial \varepsilon} \Big|_{\varepsilon=0} + \tilde{H}_1(\eta_0) \\ \quad = \int (\frac{1}{2}\eta_{0xx}\eta_1 + \frac{1}{2}\eta_0\eta_{1xx} + 3\eta_0^2\eta_1 + \frac{1}{2}\eta_0\eta_{0xx} + \eta_0^3) dx. \end{cases} \quad (4.13)$$

Noting that in this example, we have

$$K_n^{\text{per}} = K_n + \varepsilon K_n, \quad \tilde{H}_n^{\text{per}} = \tilde{H}_n + \varepsilon \tilde{H}_n, \quad n \geq 0,$$

infinitely many symmetries and conserved functionals of the triangular system (4.9) are computed as follows

$$\begin{cases} \text{per}_1(K_n^{\text{per}}) = \left[\begin{array}{c} K_n(\eta_0) \\ \frac{\partial K_n(\hat{u}_1)}{\partial \varepsilon} \Big|_{\varepsilon=0} + K_n(\eta_0) \end{array} \right], \quad n \geq 0, \\ \text{per}_1(\tilde{H}_n^{\text{per}}) = \frac{\partial \tilde{H}_n(\hat{u}_1)}{\partial \varepsilon} \Big|_{\varepsilon=0} + \tilde{H}_n(\eta_0), \quad n \geq 0. \end{cases} \quad (4.14)$$

Secondly, we take another special perturbed equation

$$u_t = K^{\text{per}}(u) = K_1(u) + \varepsilon K_2(u), \quad (4.15)$$

as a starting equation, which can be written as a bi-Hamiltonian system

$$u_t = K^{\text{per}}(u) = J \frac{\delta \tilde{H}_1^{\text{per}}}{\delta u} = M \frac{\delta \tilde{H}_0^{\text{per}}}{\delta u}, \quad \tilde{H}_0^{\text{per}} = \tilde{H}_0 + \varepsilon \tilde{H}_1, \quad \tilde{H}_1^{\text{per}} = \tilde{H}_1 + \varepsilon \tilde{H}_2. \quad (4.16)$$

Here $K_1 = K$, $J, M, \tilde{H}_0, \tilde{H}_1, \tilde{H}_2, K_2$, are determined by (4.1), (4.3) and (4.6). The second-order perturbation yields the following triangular system:

$$\begin{cases} \eta_{0t} = \eta_{0xxx} + 6\eta_0\eta_{0x}, \\ \eta_{1t} = \eta_{1xxx} + 6(\eta_0\eta_1)_x + K_2(\eta_0), \\ \eta_{2t} = \eta_{2xxx} + 6(\eta_0\eta_2)_x + 6\eta_1\eta_{1x} + \frac{\partial K_2(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0}, \end{cases} \quad (4.17)$$

where

$$\begin{aligned} \frac{\partial K_2(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} &= \eta_{1,5x} + 10\eta_1\eta_{0xxx} + 10\eta_0\eta_{1xxx} \\ &\quad + 20\eta_{1x}\eta_{0xx} + 20\eta_{0x}\eta_{1xx} + 60\eta_0\eta_{0x}\eta_1 + 30\eta_0^2\eta_{1x}. \end{aligned} \quad (4.18)$$

The corresponding Hamiltonian pair and hereditary recursion operator are

$$\hat{J}_2 = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & \partial_x & 0 \\ \partial_x & 0 & 0 \end{bmatrix}, \quad \hat{M}_2 = \begin{bmatrix} 0 & 0 & M_0 \\ 0 & M_0 & M_1 \\ M_0 & M_1 & M_2 \end{bmatrix}, \quad \hat{\Phi}_2 = \begin{bmatrix} \Phi_0 & 0 & 0 \\ \Phi_1 & \Phi_0 & 0 \\ \Phi_2 & \Phi_1 & \Phi_0 \end{bmatrix}, \quad (4.19)$$

where the entries of \hat{M}_2 and $\hat{\Phi}_2$ are given by

$$M_i = \delta_{i0}\partial_x^3 + 4\eta_0\partial_x + 2\eta_{0x}, \quad \Phi_i = \delta_{i0}\partial_x^2 + 4\eta_i + 2\eta_{ix}\partial_x^{-1}, \quad 0 \leq i \leq 2. \quad (4.20)$$

Through our scheme of construction in Section 3, the triangular system (4.17) has a local bi-Hamiltonian formulation

$$\hat{\eta}_{2t} = \hat{J}_2 \frac{\delta(\text{per}_2 \tilde{H}_1^{\text{per}})}{\delta \hat{\eta}_2} = \hat{M}_2 \frac{\delta(\text{per}_2 \tilde{H}_0^{\text{per}})}{\delta \hat{\eta}_2}, \quad \hat{\eta}_2 = (\eta_0, \eta_1, \eta_2)^T, \quad (4.21)$$

with two Hamiltonian functionals

$$\left\{ \begin{array}{l} \text{per}_2 \tilde{H}_0^{\text{per}} = (\text{per}_2 \tilde{H}_0^{\text{per}})(\hat{\eta}_2) = \frac{1}{2} \frac{\partial^2 \tilde{H}_0(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial \tilde{H}_1(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} \\ \quad = \int \left(\frac{1}{2} \eta_1^2 + \eta_0 \eta_2 + \frac{1}{2} \eta_{0xx} \eta_1 + \frac{1}{2} \eta_0 \eta_{1xx} + 3\eta_0^2 \eta_1 \right) dx, \\ \text{per}_2 \tilde{H}_1^{\text{per}} = (\text{per}_2 \tilde{H}_1^{\text{per}})(\hat{\eta}_2) = \frac{1}{2} \frac{\partial^2 \tilde{H}_1(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial \tilde{H}_2(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} \\ \quad = \int \left(\frac{1}{2} \eta_{0xx} \eta_2 + \frac{1}{2} \eta_1 \eta_{1xx} + \frac{1}{2} \eta_0 \eta_{2xx} + 3\eta_0 \eta_1^2 + 3\eta_0^2 \eta_2 \right) dx + \frac{\partial \tilde{H}_2(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0}, \end{array} \right. \quad (4.22)$$

where

$$\begin{aligned} \frac{\partial \tilde{H}_2(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} &= \int \left(\frac{1}{2} \eta_{0,4x} \eta_1 + \frac{1}{2} \eta_0 \eta_{1,4x} + \frac{5}{3} \eta_{0x}^2 \eta_1 \right. \\ &\quad \left. + \frac{10}{3} \eta_0 \eta_{0x} \eta_{1x} + \frac{20}{3} \eta_0 \eta_{0xx} \eta_1 + \frac{10}{3} \eta_0^2 \eta_{1x} + 10 \eta_0^3 \eta_1 \right) dx. \end{aligned} \quad (4.23)$$

Its infinitely many symmetries and conserved functionals read as

$$\left\{ \begin{array}{l} \text{per}_2(K_n^{\text{per}}) = \left[\begin{array}{c} K_n(\eta_0) \\ \frac{\partial K_n(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} + K_{n+1}(\eta_0), \\ \frac{1}{2} \frac{\partial^2 K_n(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial K_{n+1}(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} \end{array} \right], \quad n \geq 0, \\ \text{per}_2 \tilde{H}_n^{\text{per}} = \frac{1}{2} \frac{\partial^2 \tilde{H}_n(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial \tilde{H}_{n+1}(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0}, \quad n \geq 0, \end{array} \right. \quad (4.24)$$

by using $K_n^{\text{per}} = K_n + \varepsilon K_{n+1}$ and $\tilde{H}_n^{\text{per}} = \tilde{H}_n + \varepsilon \tilde{H}_{n+1}$ in this example.

4.2 The case of bi-scale perturbations:

We would like to exhibit two examples in the case of bi-scale perturbations and show that multi-scale perturbations can lead to bi-Hamiltonian systems in higher spatial dimensions. A concrete example in $2 + 1$ dimensions for the KdV equation is the following triangular system

$$\left\{ \begin{array}{l} \eta_{0t_1} = \eta_{0xxx} + 6\eta_0 \eta_{0x}, \\ \eta_{1t_1} = \eta_{1xxx} + 3\eta_{0xy} + 6(\eta_0 \eta_1)_x + 6\eta_0 \eta_{0y}, \end{array} \right. \quad (4.25)$$

resulting from the KdV equation (4.1) by a first-order bi-scale perturbation

$$\hat{u}_1 = \eta_0(t, x, y) + \varepsilon \eta_1(t, x, y), \quad y = \varepsilon x. \quad (4.26)$$

This systems was furnished in [16], and based on our scheme of construction in Section 3, it has the following local bi-Hamiltonian formulation

$$\hat{\eta}_{1t} = \hat{J}_1 \frac{\delta(\text{per}_1 \tilde{H}_1)}{\delta \hat{\eta}_1} = \hat{M}_1 \frac{\delta(\text{per}_1 \tilde{H}_0)}{\delta \hat{\eta}_1}, \quad \hat{\eta}_1 = (\eta_0, \eta_1)^T, \quad (4.27)$$

where the Hamiltonian pair reads as

$$\hat{J}_1 = \begin{bmatrix} 0 & \partial_x \\ \partial_x & \partial_y \end{bmatrix}, \quad \hat{M}_1 = \begin{bmatrix} 0 & \partial_x^3 + 2\eta_{0x} + 4\eta_0 \partial_x \\ \partial_x^3 + 2\eta_{0x} + 4\eta_0 \partial_x & Q \end{bmatrix}, \quad (4.28)$$

with Q being defined by

$$Q = \frac{\partial}{\partial \varepsilon} M(\hat{u}_1) \Big|_{\varepsilon=0} = 3\partial_x^2 \partial_y + 2\eta_{1x} + 2\eta_{0y} + 4\eta_1 \partial_x + 4\eta_0 \partial_y, \quad (4.29)$$

and the Hamiltonian functionals are

$$\begin{cases} \text{per}_1 \tilde{H}_0 = \frac{\partial}{\partial \varepsilon} \tilde{H}_0(\hat{u}_1) \Big|_{\varepsilon=0} = \iint \eta_0 \eta_1 \, dx dy, \\ \text{per}_1 \tilde{H}_1 = \frac{\partial}{\partial \varepsilon} \tilde{H}_1(\hat{u}_1) \Big|_{\varepsilon=0} = \iint (\frac{1}{2} \eta_0 \eta_{1xx} + \eta_0 \eta_{0xy} + \frac{1}{2} \eta_1 \eta_{0xx} + 3\eta_0^3 \eta_1) \, dx dy. \end{cases} \quad (4.30)$$

Moreover, the above Hamiltonian pair yields a hereditary recursion operator in $2+1$ dimensions

$$\hat{\Phi}_1 = \begin{bmatrix} \partial_x^2 + 2\eta_{0x} \partial_x^{-1} + 4\eta_0 & 0 \\ 2\partial_x \partial_y - 2\eta_{0x} \partial_x^{-2} \partial_y + 2(\eta_{1x} + \eta_{0y}) \partial_x^{-1} + 4\eta_1 & \partial_x^2 + 2\eta_{0x} \partial_x^{-1} + 4\eta_0 \end{bmatrix}, \quad (4.31)$$

for the triangular system (4.25).

The second example is the following

$$\begin{cases} \eta_{0t_1} &= K_1(\eta_0) = \eta_{0xxx} + 6\eta_0 \eta_{0x}, \\ \eta_{1t_1} &= \frac{\partial K_1(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} + K_1(\eta_0) \\ &= \eta_{1xxx} + 3\eta_{0xxy} + 6(\eta_0 \eta_1)_x + 6\eta_0 \eta_{0y} + \eta_{0xxx} + 6\eta_0 \eta_{0x}, \\ \eta_{2t_1} &= \frac{1}{2!} \frac{\partial^2 K_1(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial K_1(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} \\ &= \eta_{2xxx} + 3\eta_{1xxy} + 3\eta_{0xyy} + 6(\eta_0 \eta_2)_x + 6\eta_1 \eta_{1x} + 6(\eta_0 \eta_1)_y \\ &\quad + \eta_{1xxx} + 3\eta_{0xxy} + 6(\eta_0 \eta_1)_x + 6\eta_0 \eta_{0y}, \end{cases} \quad (4.32)$$

which can be generated from a perturbed KdV equation (4.7) under the second-order bi-scale perturbation

$$\hat{u}_2 = \eta(t, x, y) + \varepsilon \eta_1(t, x, y) + \varepsilon \eta_2(t, x, y), \quad y = \varepsilon x. \quad (4.33)$$

According to our scheme of construction in Section 3, the corresponding Hamiltonian pair and recursion operator read as

$$\hat{J}_2 = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & \partial_x & \partial_y \\ \partial_x & \partial_y & 0 \end{bmatrix}, \quad \hat{M}_2 = \begin{bmatrix} 0 & 0 & M_0 \\ 0 & M_0 & M_1 \\ M_0 & M_1 & M_2 \end{bmatrix}, \quad \hat{\Phi}_2 = \begin{bmatrix} \Phi_0 & 0 & 0 \\ \Phi_1 & \Phi_0 & 0 \\ \Phi_2 & \Phi_1 & \Phi_0 \end{bmatrix}, \quad (4.34)$$

where the entries of \hat{M}_2 are defined by

$$\begin{cases} M_0 = M(\hat{u}_2)|_{\varepsilon=0} = \partial_x^3 + 2\eta_{0x} + 4\eta_0\partial_x \\ M_1 = \frac{1}{1!} \frac{\partial M(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} = 3\partial_x^2\partial_y + 2\eta_{1x} + 2\eta_{0y} + 4\eta_1\partial_x + 4\eta_0\partial_y, \\ M_2 = \frac{1}{2!} \frac{\partial^2 M(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} = 3\partial_x\partial_y^2 + 2\eta_{2x} + 2\eta_{1y} + 4\eta_2\partial_x + 4\eta_1\partial_y, \end{cases} \quad (4.35)$$

and the entries of $\hat{\Phi}_2$, by

$$\begin{cases} \Phi_0 = \Phi(\hat{u}_2)|_{\varepsilon=0} = \partial_x^2 + 2\eta_{0x}\partial_x^{-1} + 4\eta_0, \\ \Phi_1 = \frac{1}{1!} \frac{\partial \Phi(\hat{u}_2)}{\partial \varepsilon} \Big|_{\varepsilon=0} = 2\partial_x\partial_y + 2(\eta_{1x} + \eta_{0y})\partial_x^{-1} - 2\eta_{0x}\partial_x^{-2}\partial_y + 4\eta_1, \\ \Phi_2 = \frac{1}{2!} \frac{\partial^2 \Phi(\hat{u}_2)}{\partial \varepsilon^2} \Big|_{\varepsilon=0} = \partial_y^2 + 2(\eta_{2x} + \eta_{1y})\partial_x^{-1} - 2(\eta_{1x} + \eta_{0y})\partial_x^{-2}\partial_y + 2\eta_{0x}\partial_x^{-3}\partial_y^2 + 4\eta_2. \end{cases} \quad (4.36)$$

The 2+1 triangular system (4.32) has a local bi-Hamiltonian formulation

$$\hat{\eta}_{2t} = \hat{J}_2 \frac{\delta(\text{per}_2 \tilde{H}_1)}{\delta \hat{\eta}_2} = \hat{M}_2 \frac{\delta(\text{per}_2 \tilde{H}_0)}{\delta \hat{\eta}_2}, \quad \hat{\eta}_2 = (\eta_0, \eta_1, \eta_2)^T, \quad (4.37)$$

with a Hamiltonian pair \hat{J}_2 and \hat{M}_2 being defined by (4.34), and two Hamiltonian functionals, by

$$\begin{cases} \text{per}_2 \tilde{H}_0^{\text{per}} = \frac{1}{2!} \frac{\partial^2 \tilde{H}_0^{\text{per}}}{\partial \varepsilon^2} \Big|_{\varepsilon=0} = \frac{1}{2!} \frac{\partial^2 \tilde{H}_0}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial \tilde{H}_0}{\partial \varepsilon} \Big|_{\varepsilon=0} = \iint (\frac{1}{2}\eta_0\eta_2 + \frac{1}{4}\eta_1^2 + \eta_0\eta_1) dx dy, \\ \text{per}_2 \tilde{H}_1^{\text{per}} = \frac{1}{2!} \frac{\partial^2 \tilde{H}_1^{\text{per}}}{\partial \varepsilon^2} \Big|_{\varepsilon=0} = \frac{1}{2!} \frac{\partial^2 \tilde{H}_1}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \frac{\partial \tilde{H}_1}{\partial \varepsilon} \Big|_{\varepsilon=0} \\ = \iint [\frac{1}{4}(\eta_0\eta_{2xx} + \eta_1\eta_{1xx} + \eta_2\eta_{0xx} + 2\eta_0\eta_{1xx} + 2\eta_1\eta_{0xx}) \\ + \frac{1}{2}(\eta_0\eta_{1xy} + \eta_1\eta_{0xy} + 2\eta_0\eta_{0xy}) + \frac{1}{4}\eta_0\eta_{0yy} + 3(\eta_0\eta_1^2 + \eta_0^2\eta_1 + \eta_0^2\eta_2)] dx dy. \end{cases} \quad (4.38)$$

Both 2 + 1 dimensional triangular systems above have infinitely many symmetries and conserved functionals due to the existence of hereditary recursion operators, and thus they are also integrable in the sense of the existence of the Abelian symmetry algebra [17]. Note that under the bi-scale perturbation

$$\hat{u}_N = \sum_{i=0}^N \varepsilon^i \eta_i(t, x, y), \quad y = \varepsilon x,$$

we have, for example,

$$\partial_x \rightarrow \partial_x + \varepsilon \partial_y, \quad \hat{u}_{Nx} \rightarrow \sum_{i=0}^N \varepsilon^i (\eta_{ix} + \varepsilon \eta_{iy}), \quad \hat{u}_{Nxx} \rightarrow \sum_{i=0}^N \varepsilon^i (\eta_{ixx} + 2\varepsilon \eta_{ixy} + \varepsilon^2 \eta_{iyy}).$$

These equalities have been used in the above deduction of bi-Hamiltonian systems in 2 + 1 dimensions.

5 Concluding remarks

We have proposed a bi-Hamiltonian formulation (3.7) for the triangular systems (2.8) resulted by perturbations around solutions of the perturbed systems. The symmetry problem can lead to a special case (1.2) of our triangular systems (2.8), which is generated by the first-order perturbation. However, the perturbation system (1.2) is a little more general than the symmetry problem itself. It is because the second component system of the perturbation system (1.2) needs to hold only for a solution of the original system $u_t = K(u)$, but the same system in the symmetry problem needs to hold for all solutions of $u_t = K(u)$. The resulting formulation gives a way to construct various integrable couplings in both lower dimensions and higher dimensions for bi-Hamiltonian systems, all of which at least possess infinitely many commuting symmetries and conserved functionals. Four illustrative examples were given for the KdV equation, which contain two $2 + 1$ dimensional local bi-Hamiltonian systems (4.27) and (4.37).

The triangular system (4.25) was first introduced in [16], whose Painlevé property and zero curvature representation were discussed by Sakovich [18]. General triangular systems resulted by multi-scale perturbations also can possess rich structures of zero curvature representations. If multi-scale perturbations are taken into account, the involved spectral parameters, denoted by μ_i , $0 \leq i \leq N$, may vary with respect to the spatial variables [18, 6], although they need to satisfy some conditions, for example,

$$\mu_{0x} = 0, \quad \mu_{ix} + \mu_{i-1,y} = 0, \quad 1 \leq i \leq N,$$

in the case of bi-scale perturbations

$$\hat{u}_N = \sum_{i=0}^N \varepsilon^i \eta_i(x, y, t) = \sum_{i=0}^N \varepsilon^i \eta_i(x, \varepsilon x, t).$$

More interestingly, our $2 + 1$ dimensional bi-Hamiltonian systems (4.27) and (4.37) are local and possess hereditary recursion operators, and thus they enjoy a different feature from known scalar integrable equations in $2 + 1$ dimensions. To our best knowledge, (4.27) and (4.37) are the first two examples of local $2 + 1$ dimensional bi-Hamiltonian systems with hereditary recursion operators. They also can provide useful information for classifying integrable systems in $2 + 1$ dimensions by the symmetry approach [19].

We remark that our general bi-Hamiltonian formulation in Section 3 can be used to establish bi-Hamiltonian formulations for a hierarchy of coupled KdV systems introduced in [20], although it does not work for the other two hierarchies of coupled KdV systems furnished in [21, 22] and [23]. Moreover, our triangular systems, especially (2.11), starting from the KdV equation, provide examples of bi-Hamiltonian systems among the integrable coupled KdV systems described by

Gürses and Karasu [24], and general triangular systems can provide new bi-Hamiltonian systems of other types, for example, bi-Hamiltonian systems of coupled fifth KdV equations and coupled modified KdV equations. Nonlinearization resulting from symmetry constraints also can be manipulated for linking our triangular systems to finite-dimensional integrable Hamiltonian systems [25].

We finally point out that our general scheme requires a bi-Hamiltonian structure of the starting system. Nevertheless, we can still make the perturbation to get triangular systems from non-Hamiltonian systems such as the KP hierarchy, and study integrable properties for the resulting triangular systems.

Acknowledgments: The work was supported by the City University of Hong Kong (SRGs 7001041, 7001178) and the Research Grants Council of Hong Kong (CERG 9040466). The author is also grateful to the referee for helpful comments.

References

- [1] F. Magri, J. Math. Phys. **19**, 1156–1162 (1978).
- [2] K. M. Case and A. M. Roos, J. Phys. A: Math. Gen. **22**, 2824–2830 (1981).
- [3] M. Lakshmanan and K. M. Tamizhmani, J. Math. Phys. **26**, 1189–1200 (1985).
- [4] W. X. Ma and B. Fuchssteiner, Chaos, Solitons & Fractals **7**, 1227–1250 (1996).
- [5] B. Fuchssteiner, in: *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations* NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences 413, ed Clarkson P A (Dordrecht: Kluwer) pp 125–138 (1993).
- [6] W. X. Ma, Methods Appl. Anal., **7**, 21–56 (2000).
- [7] I. M. Gelfand and I. Ya Dorfman, Func. Anal. Appl. **13**, 248–262 (1979).
- [8] A. S. Fokas and B. Fuchssteiner, Lett. Nuovo Cimento **28**, 299–303 (1980).
- [9] M. Blaszak, *Multi-Hamiltonian Theory of Dynamical Systems* (Berlin: Springer-Verlag, 1998).
- [10] P. J. Olver, *Applications of Lie Groups to Differential Equations* Second edition, Graduate Texts in Mathematics 107 (New York: Springer-Verlag, 1993).
- [11] B. Fuchssteiner and A. S. Fokas, Physica D **4**, 47–66 (1981).
- [12] B. Fuchssteiner, Nonlinear Analysis TMA **3**, 849–862 (1979).
- [13] W. X. Ma, J. Phys. A: Math. Gen. **23**, 2707–2716 (1990).
- [14] W. X. Ma, Science in China A **34**, 769–782 (1991).
- [15] K. M. Tamizhmani and M. Lakshmanan, J. Phys. A: Math. Gen. **16**, 3773–3782 (1983).
- [16] W. X. Ma and B. Fuchssteiner, Phys. Lett. A **213**, 49–55 (1996).
- [17] A. S. Fokas, Studies in Appl. Math. **77**, 253–299 (1987).
- [18] S. Yu Sakovich, J. Nonlinear Math. Phys. **5**, 230–233 (1998).
- [19] A. V. Mikhailov, A. B. Shabat and V. V. Sokolov, in: *What is integrability?* Springer Series in Nonlinear Dynamics, ed Zakharov V E (Berlin: Springer) pp 115–184 (1991).

- [20] W. X. Ma, J. Phys. A: Math. Gen. **31**, 7585–7591 (1998).
- [21] W. X. Ma, *On the Generalized Hamiltonian Structure of Nonlinear Evolution Equations*, Master Thesis, Academia Sinica, Beijing, 1985.
- [22] M. Antonowicz and A. P. Fordy, Physica D **28**, 345–357 (1987).
- [23] W. X. Ma and M. Pavlov, Phys. Lett. A **28**, 511–522 (1998).
- [24] M. Gürses and A. Karasu, J. Math. Phys. **39**, 2103–2111 (1998).
- [25] W. X. Ma and R. G. Zhou, Physica A **296**, 60–74 (2001).